Linear Algebra and its Applications 430 (2009) 2929–2942

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Moore–Penrose inverses of partitioned adjointable operators on Hilbert C^*-modules ☆

Qingxiang Xu

Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China

A R T I C L E   I N F O

Article history:
Received 9 August 2008
Accepted 5 January 2009
Available online 10 February 2009

Submitted by C.-K. Li

AMS classification:
15A09
46L08

Keywords:
Moore–Penrose inverse
Weighted Moore–Penrose inverse
Partitioned adjointable operator

A B S T R A C T

Let $\mathfrak{A}$ be a C^*-algebra, $H_i (i = 1, 2, 3)$ be three Hilbert-$\mathfrak{A}$ modules, $A_1 \in \mathscr{L}(H_1, H_3)$ and $A_2 \in \mathscr{L}(H_2, H_3)$, where $\mathscr{L}(H_1, H_3)$ (resp. $\mathscr{L}(H_2, H_3)$) is the set of the adjointable operators from H_1 to H_3 (resp. H_2 to H_3). For such two operators A_1 and A_2, a 1×2 partitioned operator $A = (A_1, A_2)$ can be induced by letting $A\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = A_1h_1 + A_2h_2$ for $h_i \in H_i, i = 1, 2$. In this paper, several formulae for the Moore–Penrose inverse $A^\dagger$ of A are derived, and an approach to constructing the weighted Moore–Penrose inverse from the non-weighted case is provided. In particular, the main result of Udwadia and Phohomsiri [F.E. Udwadia, P. Phohomsiri, Recursive formulas for the generalized LM-inverse of a matrix, J. Optimiz. Theory App. 131 (2006) 1–16] is generalized.

0. Introduction

Roughly speaking, a Hilbert C^*-module is an object like a Hilbert space except that the inner product is not scalar-valued, but takes its values in a C^*-algebra. The (complex) finite-dimensional spaces, the (complex) Hilbert spaces and C^*-algebras can all be regarded as Hilbert C^*-modules. Although in a general Hilbert C^*-module, a closed topologically complemented submodule may fail to be orthogonally complemented, this deficit can be mended for the null space of an adjointable operator with closed range [5, Theorem 3.2]. This allows us, as in the Hilbert space case, to study the Moore–Penrose inverse of the adjointable operators with closed ranges [9].

☆ Partially supported by Leading Academic Discipline Project of Shanghai Normal University under Grant DZL803 and Shanghai Municipal Education Commission under Grant 09YZ147.

E-mail address: qxxu@shnu.edu.cn

0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.01.003

In this paper, we study the representations for the Moore–Penrose inverse $A^{\dagger}$ and the weighted Moore–Penrose inverse $A^{\dagger}_{MN}$ of a partitioned adjointable operator $A = (A_1, A_2)$ acting on Hilbert C^*-modules. For matrices, some formulae such as Cline [3] and Mihalyffy [6] are known. Note that the Moore–Penrose inverse of a matrix A exists uniquely, so the various representations for $A^{\dagger}$ are essentially the same. Yet it might be a tough work to check whether a given matrix B is equal to $A^{\dagger}$ or not. One usual way to be undertaken is to verify directly whether the four conditions imposed on $A^{\dagger}$ are satisfied [3,4,6]; another way often undertaken is to check whether B will be a solution of a certain matrix equation, whose solution is unique, namely $A^{\dagger}$ [2].

The purpose of this paper is not only to generalize some previous formulae for $A^{\dagger}$ and $A^{\dagger}_{MN}$ known for matrices to the case of Hilbert C^*-modules, but most of all, to provide a new approach to the representations for the generalized inverses of partitioned matrices, which are still being intensively studied. The paper is organized as follows. In Section 1, we will recall some basic knowledge about Hilbert C^*-modules. Some auxiliary results are given in Section 2, while some equalities concerning the values of the associated operators are presented in Section 3. In Section 4, various representations for $A^{\dagger}$ are given. In Section 5, an approach to constructing the weighted Moore–Penrose inverse from the non-weighted case is provided, a new representation for $A^{\dagger}_{MN}$ is given (see Theorem 5.1), and the main result of [7] is generalized.

1. Preliminaries

Throughout this paper, $\mathfrak{A}$ is a C^*-algebra, $\mathbb{C}$ is the complex field, and $\mathbb{C}^{m\times n}$ is the set of all $m \times n$ complex matrices. By a projection, we mean an idempotent and self-adjoint element in a certain C^*-algebra. Let H and K be two Hilbert $\mathfrak{A}$-modules, denote by $\mathscr{L}(H,K)$ the set of the *adjointable* operators from H to K. In case $H = K$, $\mathscr{L}(H,H)$ which we abbreviate to $\mathscr{L}(H)$, is a C^*-algebra, whose unit is denoted by I_H. For any $A \in \mathscr{L}(H,K)$, the range and the null space of A are denoted by $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively.

The notations of "$\oplus$" and "$\dot{+}$" are used in this paper with different meanings. For any Hilbert $\mathfrak{A}$-modules H_1 and H_2, let

$$H_1 \oplus H_2 = \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \middle| \, h_i \in H_i,\ i = 1,2 \right\},$$

which is also a Hilbert $\mathfrak{A}$-module whose $\mathfrak{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle \quad \text{for } x_i \in H_1 \text{ and } y_i \in H_2,\ i = 1,2.$$

If H_1 and H_2 both are submodules of a Hilbert $\mathfrak{A}$-module H such that $H_1 \cap H_2 = \{0\}$, then we define

$$H_1 \dot{+} H_2 = \{h_1 + h_2 | h_i \in H_i,\ i = 1,2\} \subseteq H.$$

Lemma 1.1 (*cf.* [5, Theorem 3.2] and [9, Remark 1.1]). *Let $A \in \mathscr{L}(H,K)$. Then the closeness of any one of the following sets implies the closeness of the remaining three sets*:

$$\mathscr{R}(A), \quad \mathscr{R}(A^*), \quad \mathscr{R}(AA^*), \quad \mathscr{R}(A^*A).$$

If $\mathscr{R}(A)$ is closed, then $\mathscr{R}(A) = \mathscr{R}(AA^*), \mathscr{R}(A^*) = \mathscr{R}(A^*A)$ and the following orthogonal decompositions hold:

$$H = \mathscr{N}(A) \dot{+} \mathscr{R}(A^*), \quad K = \mathscr{R}(A) \dot{+} \mathscr{N}(A^*). \tag{1.1}$$

Definition 1.1. Let H, K be two Hilbert $\mathfrak{A}$-modules, and $A \in \mathscr{L}(H,K)$. The Moore–Penrose inverse $A^{\dagger}$ of A (if it exists) is an element X of $\mathscr{L}(K,H)$ which satisfies

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX \quad \text{and} \quad (XA)^* = XA. \tag{1.2}$$

Remark 1.1. For any $A \in \mathscr{L}(H,K)$, as in the Hilbert space case, the Moore–Penrose inverse $A^{\dagger}$ of A exists if and only if A has closed range [9, Theorem 2.2]. In which case, $A^{\dagger}$ exists uniquely with

$$A^{\dagger}AA^{*}=A^{*} \quad \text{and} \quad A^{\dagger}|_{\mathscr{R}(A)^{\perp}}=0, \tag{1.3}$$

where $A^{\dagger}|_{\mathscr{R}(A)^{\perp}}$ is the restriction of $A^{\dagger}$ to the orthogonal complement of $\mathscr{R}(A)$.

Recall that an element a of a C^*-algebra $\mathfrak{B}$ is said to be *hermitian* if $a=a^*$; and *positive*, written $a\geq 0$, if it is hermitian and its spectrum $sp(a)$ lies in $[0,+\infty)$. It is known that a is positive if and only if $a=b^*b$ for some $b\in\mathfrak{B}$. In the special case that $\mathfrak{B}=\mathscr{L}(H)$ for some Hilbert $\mathfrak{A}$-module H, we have the following lemma:

Lemma 1.2 (*cf.* [5, Lemma 4.1]). *Let H be a Hilbert $\mathfrak{A}$-module and $T\in\mathscr{L}(H)$. The following conditions are equivalent*:

(i) *T is a positive element of $\mathscr{L}(H)$*;
(ii) *$\langle T\xi,\xi\rangle\geq 0$, for any $\xi\in H$.*

Lemma 1.3. *Let H and K be two Hilbert $\mathfrak{A}$-modules, and $A\in\mathscr{L}(H,K)$. Then I_H+A^*A and I_K+AA^* are invertible in $\mathscr{L}(H)$ and $\mathscr{L}(K)$ respectively, and*

$$A(I_H+A^*A)^{-1}=(I_K+AA^*)^{-1}A, \tag{1.4}$$
$$(I_H+A^*A)^{-1}=I_H-A^*(I_K+AA^*)^{-1}A. \tag{1.5}$$

Proof. By Lemma 1.2 we know that $A^*A\in\mathscr{L}(H)$ and $AA^*\in\mathscr{L}(K)$ both are positive, so I_H+A^*A and I_K+AA^* are invertible. Clearly, we have $A(I_H+A^*A)=(I_K+AA^*)A$, so (1.4) holds. Moreover,

$$\begin{aligned}&(I_H-A^*(I_K+AA^*)^{-1}A)(I_H+A^*A)\\&\quad=I_H+A^*A-A^*[(I_K+AA^*)^{-1}+(I_K+AA^*)^{-1}AA^*]A\\&\quad=I_H+A^*A-A^*A=I_H.\end{aligned}$$

Similarly, we can prove that $(I_H+A^*A)(I_H-A^*(I_K+AA^*)^{-1}A)=I_H$, so (1.5) also holds. $\square$

Remark 1.2. Let H and K be two Hilbert $\mathfrak{A}$-modules, $A\in\mathscr{L}(H,K)$ and $B\in\mathscr{L}(K,H)$. Suppose that $I_H+BA\in\mathscr{L}(H)$ is invertible. Then as in the proof of (1.5) we can prove that $I_K+AB\in\mathscr{L}(K)$ is also invertible with

$$(I_K+AB)^{-1}=I_K-A(I_H+BA)^{-1}B. \tag{1.6}$$

Throughout the rest of this paper, H_1,H_2 and H_3 are three Hilbert $\mathfrak{A}$-modules. Given $A_1\in\mathscr{L}(H_1,H_3)$, $A_2\in\mathscr{L}(H_2,H_3)$ such that $A_1^{\dagger}$ and $A_2^{\dagger}$ exist, let $A=(A_1,A_2)\in\mathscr{L}(H_1\oplus H_2,H_3)$ be the partitioned operator defined by

$$A\begin{pmatrix}h_1\\h_2\end{pmatrix}=A_1h_1+A_2h_2 \quad \text{for } \begin{pmatrix}h_1\\h_2\end{pmatrix}\in H_1\oplus H_2. \tag{1.7}$$

For $i=1,2$, let $P_i,Q_i\in\mathscr{L}(H_3)$ be the associated projections defined by

$$P_1=A_1A_1^{\dagger},\quad P_2=A_2A_2^{\dagger},\quad Q_1=I_{H_3}-P_1,\quad Q_2=I_{H_3}-P_2. \tag{1.8}$$

It is easy to show (see the proof of [8, Theorem 2.1] for example) that

$$A^{\dagger} \text{ exists} \iff (Q_2A_1)^{\dagger} \text{ exists} \iff (Q_1A_2)^{\dagger} \text{ exists.}$$

So in the rest of this paper, we always assume that $A_1^{\dagger},A_2^{\dagger},(Q_1A_2)^{\dagger}$ and $(Q_2A_1)^{\dagger}$ all exist. Since $\mathscr{R}(P_2)\subseteq\mathscr{R}(Q_2A_1)^{\perp}$, by (1.3) we have $(Q_2A_1)^{\dagger}P_2=0$, which means that

$$(Q_2A_1)^{\dagger}=(Q_2A_1)^{\dagger}Q_2. \text{ Similarly, } (Q_1A_2)^{\dagger}=(Q_1A_2)^{\dagger}Q_1. \tag{1.9}$$

2. Some auxiliary results

To simplify the proofs, in this section we introduce some notation. We begin with an observation (Proposition 2.1), by which we obtain two auxiliary results (Propositions 2.2 and 2.3). In addition, we will give an alternative expression for $A^\dagger$ (see (2.12)) in terms of the operators T_1 and T_2, whose concrete expressions were originally given in [8] in the Hilbert space case.

Let $A = (A_1, A_2)$ be the partitioned operator of $\mathscr{L}(H_1 \oplus H_2, H_3)$ defined by (1.7). As in [8], we define two operators $P_{H_1} \in \mathscr{L}(H_1 \oplus H_2, H_1)$ and $P_{H_2} \in \mathscr{L}(H_1 \oplus H_2, H_2)$ by

$$P_{H_1}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1 \text{ and } P_{H_2}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_2 \quad \text{for } h_1 \in H_1 \text{ and } h_2 \in H_2. \tag{2.1}$$

Clearly, we have

$$A^\dagger = \begin{pmatrix} P_{H_1}A^\dagger \\ P_{H_2}A^\dagger \end{pmatrix} \in \mathscr{L}(H_3, H_1 \oplus H_2). \tag{2.2}$$

Proposition 2.1. *Let $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in \mathscr{L}(H_3, H_1 \oplus H_2)$ with $B_1 \in \mathscr{L}(H_3, H_1)$ and $B_2 \in \mathscr{L}(H_3, H_2)$. Then $B = A^\dagger$ if and only if the following four conditions are satisfied*:

(i) $B_1|_{\mathscr{R}(A)^\perp} = 0$;
(ii) $B_1(A_1A_1^* + A_2A_2^*)h_3 = A_1^*h_3$, *for any* $h_3 \in H_3$;
(iii) $B_2|_{\mathscr{R}(A)^\perp} = 0$;
(iv) $B_2(A_1A_1^* + A_2A_2^*)h_3 = A_2^*h_3$, *for any* $h_3 \in H_3$.

Proof. Consider the orthogonal decomposition:

$$H_3 = \mathscr{R}(A) \dot{+} \mathscr{R}(A)^\perp = \mathscr{R}(AA^*) \dot{+} \mathscr{R}(A)^\perp$$

and make the observation:

$$A^* = \begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix}, \quad AA^* = A_1A_1^* + A_2A_2^*.$$

Since $A^\dagger|_{\mathscr{R}(A)^\perp} = 0$, by (2.2) we have $(P_{H_1}A^\dagger)|_{\mathscr{R}(A)^\perp} = 0$ and $(P_{H_2}A^\dagger)|_{\mathscr{R}(A)^\perp} = 0$. Moreover, for any $h_3 \in H_3$, we have

$$(P_{H_1}A^\dagger)(A_1A_1^* + A_2A_2^*)h_3 = P_{H_1}A^\dagger AA^*h_3 = P_{H_1}A^*h_3 = P_{H_1}\begin{pmatrix} A_1^*h_3 \\ A_2^*h_3 \end{pmatrix} = A_1^*h_3.$$

Similarly, $(P_{H_2}A^\dagger)(A_1A_1^* + A_2A_2^*)h_3 = A_2^*h_3$. In view of $Bh_3 = \begin{pmatrix} B_1h_3 \\ B_2h_3 \end{pmatrix}$ for $h_3 \in H_3$, the conclusion follows from the orthogonal decomposition: $H_3 = \mathscr{R}(A_1A_1^* + A_2A_2^*) \dot{+} \mathscr{R}(A)^\perp$. $\square$

Remark 2.1. Note that

$$\mathscr{R}(A)^\perp = \mathscr{N}(A^*) = \mathscr{N}\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix} = \mathscr{N}(A_1^*) \cap \mathscr{N}(A_2^*) = \mathscr{R}(A_1)^\perp \cap \mathscr{R}(A_2)^\perp, \tag{2.3}$$

so by (1.3) we know that the restrictions of $A_1^\dagger$ and $A_2^\dagger$ to $\mathscr{R}(A)^\perp$ both are identically zero, which implies that if $B_1 \in \mathscr{L}(H_3, H_1)$ is given such that conditions (i) and (ii) in Proposition 2.1 are satisfied, then $B_2 = A_2^\dagger - A_2^\dagger A_1 B_1 \in \mathscr{L}(H_3, H_2)$ will satisfy conditions (iii) and (iv). Conversely, if $B_2 \in \mathscr{L}(H_3, H_2)$ is given such that conditions (iii) and (iv) are satisfied, then $B_1 = A_1^\dagger - A_1^\dagger A_2 B_2$ will satisfy conditions (i) and (ii). Since the Moore–Penrose inverse of A is unique, we have the following two propositions:

Proposition 2.2. *Let $B_1 \in \mathscr{L}(H_3, H_1)$ be given such that $B_1|_{\mathscr{R}(A)^\perp} = 0$, and $B_1(A_1A_1^* + A_2A_2^*)h_3 = A_1^*h_3$, for any $h_3 \in H_3$. Then*

$$A^\dagger = \begin{pmatrix} B_1 \\ A_2^\dagger - A_2^\dagger A_1 B_1 \end{pmatrix}. \tag{2.4}$$

Proposition 2.3. *Let* $B_2 \in \mathscr{L}(H_3, H_2)$ *be given such that* $B_2|_{\mathscr{R}(A)^\perp} = 0$, *and* $B_2(A_1A_1^* + A_2A_2^*)h_3 = A_2^*h_3$, *for any* $h_3 \in H_3$. *Then*

$$A^\dagger = \begin{pmatrix} A_1^\dagger - A_1^\dagger A_2 B_2 \\ B_2 \end{pmatrix}. \tag{2.5}$$

To seek the above B_1 and B_2 and simplify the proofs, it is helpful to make notation as follows:

$$T_1 = P_{H_1}A^\dagger - (Q_2A_1)^\dagger \in \mathscr{L}(H_3, H_1), \tag{2.6}$$

$$T_2 = P_{H_2}A^\dagger - (Q_1A_2)^\dagger \in \mathscr{L}(H_3, H_2), \tag{2.7}$$

$$S_1 = A_1^\dagger - A_1^\dagger A_2(Q_1A_2)^\dagger \in \mathscr{L}(H_3, H_1), \tag{2.8}$$

$$S_2 = A_2^\dagger - A_2^\dagger A_1(Q_2A_1)^\dagger \in \mathscr{L}(H_3, H_2), \tag{2.9}$$

$$K_1 = A_1^\dagger A_2 \cdot (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2)) \in \mathscr{L}(H_2, H_1), \tag{2.10}$$

$$K_2 = A_2^\dagger A_1 \cdot (I_{H_1} - (Q_2A_1)^\dagger(Q_2A_1)) \in \mathscr{L}(H_1, H_2). \tag{2.11}$$

By the definitions of $T_i, S_i, K_i (i = 1, 2)$ and (1.9) we have

$$A^\dagger = \begin{pmatrix} P_{H_1}A^\dagger \\ P_{H_2}A^\dagger \end{pmatrix} = \begin{pmatrix} (Q_2A_1)^\dagger + T_1 \\ (Q_1A_2)^\dagger + T_2 \end{pmatrix} \tag{2.12}$$

and

$$K_1 = S_1A_2, \quad K_2 = S_2A_1, \tag{2.13}$$

$$(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))K_1^* = K_1^*, \tag{2.14}$$

$$(I_{H_1} - (Q_2A_1)^\dagger(Q_2A_1))K_2^* = K_2^*, \tag{2.15}$$

$$K_1(A_1^\dagger A_2)^* = K_1K_1^*, \quad K_2(A_2^\dagger A_1)^* = K_2K_2^*. \tag{2.16}$$

The concrete expressions for T_1 and T_2 can be described as follows:

Lemma 2.4 (*cf.* [8, Theorem 2.2]). *The restrictions of* T_1 *and* T_2 *to* $\mathscr{R}(A)^\perp$ *both are identically zero, and for any* $h_3 \in H_3$,

$$T_1((A_1A_1^* + A_2A_2^*)h_3) = (I_{H_1} - (Q_2A_1)^\dagger(Q_2A_1))A_1^*h_3, \tag{2.17}$$

$$T_2((A_1A_1^* + A_2A_2^*)h_3) = (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*h_3. \tag{2.18}$$

Lemma 2.5 (*cf.* [8, Proposition 2.1]). *Let* $T_i, S_i (i = 1, 2)$ *be defined by* (2.6)–(2.9), *respectively. Then*

$$P_{H_1}A^\dagger = S_1 - A_1^\dagger A_2 T_2, \quad P_{H_2}A^\dagger = S_2 - A_2^\dagger A_1 T_1. \tag{2.19}$$

3. Some equalities

In this section, we will derive some equalities concerning the values of the operators K_1 and K_2 defined in the previous section.

Lemma 3.1. *Let* T_i, K_i $(i = 1, 2)$ *be defined by* (2.6), (2.7), (2.10) *and* (2.11), *respectively. Then for any* $h_3 \in H_3$, *we have*

$$K_1^*A_1^*h_3 = T_2(A_1A_1^* + A_2A_2^*)h_3, \quad K_2^*A_2^*h_3 = T_1(A_1A_1^* + A_2A_2^*)h_3. \tag{3.1}$$

Proof. Since $(A_1^\dagger)^*A_1^* = A_1A_1^\dagger, A_2^*Q_1 = (Q_1A_2)^*$ and $(Q_1A_2)^\dagger(Q_1A_2)(Q_1A_2)^* = (Q_1A_2)^*$, by (2.18) we know that for any $h_3 \in H_3$,

$$\begin{aligned}K_1^*A_1^*h_3 &= (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*(A_1^\dagger)^*A_1^*h_3 = (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*A_1A_1^\dagger h_3\\ &= (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*(I_{H_3} - Q_1)h_3 = (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*h_3\\ &= T_2(A_1A_1^* + A_2A_2^*)h_3.\end{aligned}$$

The right hand equality in (3.1) can be proved similarly. □

Lemma 3.2. *Let S_i, K_i $(i = 1, 2)$ be defined by (2.8)–(2.11), respectively. Then for any $h_3 \in H_3$, we have*

$$(I_{H_1} + K_1K_1^*)A_1^*h_3 = S_1(A_1A_1^* + A_2A_2^*)h_3, \tag{3.2}$$

$$(I_{H_2} + K_2K_2^*)A_2^*h_3 = S_2(A_1A_1^* + A_2A_2^*)h_3. \tag{3.3}$$

Proof. Since $I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2)$ is a projection, $(Q_1A_2)^\dagger = (Q_1A_2)^\dagger Q_1$ and $Q_1A_1 = 0$, by (2.18) and Lemma 3.1 we know that for any $h_3 \in H_3$,

$$\begin{aligned}(I_{H_1} + K_1K_1^*)A_1^*h_3 &= A_1^*h_3 + K_1(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*h_3\\ &= A_1^*h_3 + A_1^\dagger A_2(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))A_2^*h_3\\ &= A_1^*h_3 + A_1^\dagger A_2(I_{H_2} - (Q_1A_2)^\dagger A_2)A_2^*h_3\\ &= A_1^*h_3 + A_1^\dagger A_2A_2^*h_3 - A_1^\dagger A_2(Q_1A_2)^\dagger A_2A_2^*h_3\\ &= (A_1^\dagger - A_1^\dagger A_2(Q_1A_2)^\dagger))(A_1A_1^* + A_2A_2^*)h_3\\ &= S_1(A_1A_1^* + A_2A_2^*)h_3.\end{aligned}$$

The proof of (3.3) is similar. □

Lemma 3.3. *Let K_1, K_2 be defined by (2.10) and (2.11), respectively. Then*

$$(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))(I_{H_2} + K_1^*K_1)^{-1}(A_1^\dagger A_2)^* = K_1^*(I_{H_1} + K_1K_1^*)^{-1},$$

$$(I_{H_1} - (Q_2A_1)^\dagger(Q_2A_1))(I_{H_1} + K_2^*K_2)^{-1}(A_2^\dagger A_1)^* = K_2^*(I_{H_2} + K_2K_2^*)^{-1}.$$

Proof. By (2.14) we have

$$(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))(I_{H_2} + K_1^*K_1) = I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2) + K_1^*K_1,$$

so

$$I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2) = (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))(I_{H_2} + K_1^*K_1) - K_1^*K_1,$$

hence by (1.5) in Lemma 1.3 we have

$$\begin{aligned}&(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))(I_{H_2} + K_1^*K_1)^{-1}\\ &\quad= I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2) - K_1^*K_1(I_{H_2} + K_1^*K_1)^{-1}\\ &\quad= (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2)) - (I_{H_2} - (I_{H_2} + K_1^*K_1)^{-1})\\ &\quad= (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2)) - K_1^*(I_{H_1} + K_1K_1^*)^{-1}K_1.\end{aligned}$$

It follows from (2.16) that

$$\begin{aligned}&(I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))(I_{H_2} + K_1^*K_1)^{-1}(A_1^\dagger A_2)^*\\ &\quad= K_1^* - K_1^*(I_{H_1} + K_1K_1^*)^{-1}K_1K_1^*\\ &\quad= K_1^*(I_{H_1} - (I_{H_1} + K_1K_1^*)^{-1}K_1K_1^*) = K_1^*(I_{H_1} + K_1K_1^*)^{-1}.\end{aligned}$$

Replacing A_1, K_1 with A_2 and K_2, respectively, the second equation of this lemma holds. □

4. Some formulae for the Moore–Penrose inverse $(A_1, A_2)^\dagger$

Theorem 4.1 (Mihalyffy (1)). *Let* S_1, K_1 *be defined by* (2.8) *and* (2.10), *respectively*. *Then*

$$A^\dagger = \begin{pmatrix} (I_{H_1} + K_1K_1^*)^{-1}S_1 \\ K_1^*(I_{H_1} + K_1K_1^*)^{-1}S_1 + (Q_1A_2)^\dagger \end{pmatrix}. \tag{4.1}$$

Proof. Let

$$B_1 = (I_{H_1} + K_1K_1^*)^{-1}S_1, \quad C_2 = K_1^*B_1 \quad \text{and} \quad B_2 = C_2 + (Q_1A_2)^\dagger.$$

Note that by (2.3) we have $\mathscr{R}(A)^\perp = \mathscr{R}(A_1)^\perp \cap \mathscr{N}(A_2^*)$, so if $h_3 \in \mathscr{R}(A)^\perp$, then $A_2^*h_3 = 0$ and by (1.3) we have $A^\dagger h_3 = 0$ and $A_1^\dagger h_3 = 0$, which means that $A_2^*Q_1h_3 = A_2^*h_3 - A_2^*A_1A_1^\dagger h_3 = 0$. It follows that

$$h_3 \in \mathscr{R}(Q_1A_2)^\perp \Longrightarrow (Q_1A_2)^\dagger h_3 = 0 \Longrightarrow S_1h_3 = A_1^\dagger h_3 - A_1^\dagger A_2(Q_1A_2)^\dagger h_3 = 0.$$

Therefore, $S_1|_{\mathscr{R}(A)^\perp} = 0$, hence $B_1|_{\mathscr{R}(A)^\perp} = 0$. Moreover, by Lemma 3.2 we have

$$B_1(A_1A_1^* + A_2A_2^*)h_3 = A_1^*h_3 \quad \text{for any } h_3 \in H_3,$$

so B_1 satisfies conditions (i) and (ii) stated in Proposition 2.1. We may combine Lemmas 3.1 and 3.2 to prove that $C_2 = T_2$, the conclusion then follows from (2.12). $\square$

In view of Propositions 2.2 and 2.3, and Theorem 4.1, other Mihalyffy formulae can also be obtained. We choose two of them as follows:

Theorem 4.2 (Mihalyffy (2)). *Let* S_1, K_1 *be defined by* (2.8) *and* (2.10), *respectively*. *Then*

$$A^\dagger = \begin{pmatrix} (I_{H_1} + K_1K_1^*)^{-1}S_1 \\ A_2^\dagger - A_2^\dagger A_1(I_{H_1} + K_1K_1^*)^{-1}S_1 \end{pmatrix}. \tag{4.2}$$

Theorem 4.3 (Mihalyffy (3)). *Let* S_2, K_2 *be defined by* (2.9) *and* (2.11), *respectively*. *Then*

$$A^\dagger = \begin{pmatrix} K_2^*(I_{H_2} + K_2K_2^*)^{-1}S_2 + (Q_2A_1)^\dagger \\ (I_{H_2} + K_2K_2^*)^{-1}S_2 \end{pmatrix}. \tag{4.3}$$

We may combine Theorem 4.1, Lemma 3.3 and Propositions 2.2 and 2.3 to get two Cline formulae for $A^\dagger$ as follows:

Theorem 4.4 (Cline (1)). *Let* S_1, K_1 *be defined by* (2.8) *and* (2.10), *respectively*. *Then*

$$A^\dagger = \begin{pmatrix} A_1^\dagger - A_1^\dagger A_2B_2 \\ B_2 \end{pmatrix}, \tag{4.4}$$

where $B_2 = (I_{H_2} - (Q_1A_2)^\dagger(Q_1A_2))(I_{H_2} + K_1^*K_1)^{-1}(A_1^\dagger A_2)^*S_1 + (Q_1A_2)^\dagger$.

Theorem 4.5 (Cline (2)). *Let* S_2, K_2 *be defined by* (2.9) *and* (2.11), *respectively*. *Then*

$$A^\dagger = \begin{pmatrix} B_1 \\ A_2^\dagger - A_2^\dagger A_1B_1 \end{pmatrix}, \tag{4.5}$$

where $B_1 = (I_{H_1} - (Q_2A_1)^\dagger(Q_2A_1))(I_{H_1} + K_2^*K_2)^{-1}(A_2^\dagger A_1)^*S_2 + (Q_2A_1)^\dagger$.

Theorem 4.6. *Let* S_1, K_1 *be defined by* (2.8) *and* (2.10), *respectively*. *Then*

$$A^\dagger = \begin{pmatrix} A_1^\dagger - A_1^\dagger A_2B_2 \\ B_2 \end{pmatrix}, \tag{4.6}$$

where $B_2 = (I_{H_2} + K_1^*A_1^\dagger A_2)^{-1}(K_1^*A_1^\dagger + (Q_1A_2)^\dagger)$.

Proof. Let $X = K_1^*$ and $Y = A_1^\dagger A_2$. Then $I_{H_1} + YX = I_{H_1} + K_1K_1^*$, so $I_{H_2} + XY = I_{H_2} + K_1^*A_1^\dagger A_2$ is invertible by (1.6) in Remark 1.2. For any $h_3 \in H_3$, since $(Q_1A_2)^\dagger A_1 = 0$, by (3.1) and (2.18) we have,

$$(K_1^*A_1^\dagger + (Q_1A_2)^\dagger)(A_1A_1^* + A_2A_2^*)h_3 = (I_{H_2} + K_1^*A_1^\dagger A_2)A_2^*h_3.$$

It follows that $B_2 = P_{H_2}A^\dagger$, the conclusion then follows from Proposition 2.3. □

Remark 4.1. The finite-dimensional case of the preceding formula can be found in [4]. The proof given here is quite different from that of [4]. Alternatively, we have another formula as follows:

Theorem 4.7. *Let S_2, K_2 be defined by (2.9) and (2.11), respectively. Then*

$$A^\dagger = \begin{pmatrix} B_1 \\ A_2^\dagger - A_2^\dagger A_1 B_1 \end{pmatrix}, \tag{4.7}$$

where $B_1 = (I_{H_1} + K_2^*A_2^\dagger A_1)^{-1}(K_2^*A_2^\dagger + (Q_2A_1)^\dagger)$.

Remark 4.2. In the finite-dimensional case, a result of Baksalary and Baksalary [1] indicates that $A^\dagger = \begin{pmatrix} (Q_2A_1)^\dagger \\ (Q_1A_2)^\dagger \end{pmatrix}$ if and only if $R(A_1) \cap R(A_2) = \{0\}$, if and only if $A^\dagger = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$. As shown in the Hilbert space case [8], such a result can be extended further in the context of Hilbert C^*-modules.

5. A new approach to the study of the weighted Moore–Penrose inverse $(A_1, A_2)^\dagger_{MN}$

The weighted Moore–Penrose inverse of a matrix, or a bounded linear operator on a Hilbert space, has been intensely studied by many mathematicians. The general theory of the weighted Moore–Penrose inverse can be generalized directly from the Hilbert space case to the Hilbert C^*-module case. In this section, we will, in the general setting of Hilbert C^*-module operators, provide an approach to construct the weighted Moore–Penrose inverse from the non-weighted case. Our key point is the construction of a commutative diagram on page 12. We take Theorem 4.6 as an example to illustrate how this approach works. The main result of this section is Theorem 5.1, which as far as we know, is new even for matrices. As a result, the main result of [7] is generalized. Throughout this section, H, K, H_1, H_2 and H_3 are Hilbert $\mathfrak{A}$-modules.

Definition 5.1. An element M of $\mathscr{L}(K)$ is said to be *positive definite*, if M is positive and invertible in $\mathscr{L}(K)$.

Suppose that $M \in \mathscr{L}(K)$ is positive definite. Then a new inner-product on K can be given by letting

$$\langle x, y\rangle_M = \langle x, My\rangle \quad \text{for any } x, y \in K. \tag{5.1}$$

It is easy to verify that with the above inner-product, K also becomes a Hilbert $\mathfrak{A}$-module.

Remark 5.1. The Hilbert $\mathfrak{A}$-module with the inner-product given by (5.1) is denoted by K_M, and is called *the weighted space*. Similarly, any positive definite element N of $\mathscr{L}(H)$ can induce a Hilbert $\mathfrak{A}$-module H_N. Now given any $T \in \mathscr{L}(H, K)$, T can also be regarded as an operator from H_N to K_M. If we use the notation $T^\# \in \mathscr{L}(K_M, H_N)$ to denote the adjointable operator of $T \in \mathscr{L}(H_N, K_M)$, then since for any $x \in H$ and $y \in K$,

$$\langle Tx, y\rangle_M = \langle Tx, My\rangle = \langle x, T^*My\rangle = \langle x, N^{-1}T^*My\rangle_N,$$

we have

$$T^\# = N^{-1}T^*M. \tag{5.2}$$

Definition 5.2. Let $A \in \mathscr{L}(H,K)$ be arbitrary, $M \in \mathscr{L}(K)$ and $N \in \mathscr{L}(H)$ be positive definite. The *weighted Moore–Penrose inverse* $A^{\dagger}_{MN}$ (if it exists) is the unique element X of $\mathscr{L}(K,H)$, which satisfies

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX \quad \text{and} \quad (NXA)^* = NXA. \tag{5.3}$$

It can be deduced easily from [9, Theorem 2.2] that $A^{\dagger}_{MN}$ exists if and only if A has closed range. Furthermore, by (5.2) we have

$$(A^{\dagger}_{MN}A)^* = NA^{\dagger}_{MN}AN^{-1}, \quad (AA^{\dagger}_{MN})^* = MAA^{\dagger}_{MN}M^{-1}. \tag{5.4}$$

5.1. The unitary equivalence of Hilbert C^*-modules induced by a projection

For any Hilbert $\mathfrak{A}$-module X, any projection P of $\mathscr{L}(X)$, let $X_1 = PX$ and $X_2 = (I_X - P)X$, and define $\lambda_X : X \to X_1 \oplus X_2$ by

$$\lambda_X(x) = \begin{pmatrix} Px \\ x - Px \end{pmatrix} \quad \text{for } x \in X. \tag{5.5}$$

Then λ_X is a unitary operator with $\lambda_X^* = \lambda_X^{-1}$, where λ_X^{-1} is given by

$$\lambda_X^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2 \quad \text{for } x_i \in X_i, \ i = 1,2.$$

Now let H_1 and H_2 be two Hilbert $\mathfrak{A}$-modules,

$$N = \begin{pmatrix} N_1 & L \\ L^* & N_2 \end{pmatrix} \in \mathscr{L}(H_1 \oplus H_2) \tag{5.6}$$

be a positive definite element, where $N_1 \in \mathscr{L}(H_1), L \in \mathscr{L}(H_2,H_1)$ and $N_2 \in \mathscr{L}(H_2)$. Let $S(N) = N_2 - L^*N_1^{-1}L$ be the Schur complement of N_1 in N. It is easy to verify that

$$\begin{pmatrix} I_{H_1} & 0 \\ -L^*N_1^{-1} & I_{H_2} \end{pmatrix} N \begin{pmatrix} I_{H_1} & -N_1^{-1}L \\ 0 & I_{H_2} \end{pmatrix} = \operatorname{diag}(N_1, S(N)) \stackrel{def}{=} W, \tag{5.7}$$

so $S(N) \in \mathscr{L}(H_2)$ is also positive definite. Define

$$a = N_1^{-1}L \quad \text{and} \quad P = \begin{pmatrix} I_{H_1} & a \\ 0 & 0 \end{pmatrix}. \tag{5.8}$$

Then $P^2 = P$ and $NP = P^*N$, so $P^{\#} = N^{-1}P^*N = P$, which means that P is a projection of $\mathscr{L}(X)$, where $X = (H_1 \oplus H_2)_N$ is the weighted Hilbert $\mathfrak{A}$-module whose inner-product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_N = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, N\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, N_1x_2 + Ly_2\rangle + \langle y_1, L^*x_2 + N_2y_2\rangle$$

for $x_i \in H_1$ and $y_i \in H_2, i = 1,2$. By (5.8), we have

$$X_1 = PX = \left\{ \begin{pmatrix} h_1 + ah_2 \\ 0 \end{pmatrix} \middle| h_i \in H_i \right\} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \middle| u \in H_1 \right\}, \tag{5.9}$$

$$X_2 = (I_X - P)X = \left\{ \begin{pmatrix} -ah_2 \\ h_2 \end{pmatrix} \middle| h_2 \in H_2 \right\}. \tag{5.10}$$

With the inner products inherited from X, X_1 and X_2 both are Hilbert $\mathfrak{A}$-modules. Let $j_{H_1} : (H_1)_{N_1} \to X_1$ and $j_{H_2} : (H_2)_{S(N)} \to X_2$ be defined by

$$j_{H_1}(h_1) = \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \text{ and } j_{H_2}(h_2) = \begin{pmatrix} -ah_2 \\ h_2 \end{pmatrix} \quad \text{for } h_i \in H_i, i = 1,2.$$

It is easy to verify that j_{H_1} and j_{H_2} both are invertible with

$$j_{H_1}^{-1}\begin{pmatrix} h_1 \\ 0 \end{pmatrix} = h_1 \ \text{ and } \ j_{H_2}^{-1}\begin{pmatrix} -ah_2 \\ h_2 \end{pmatrix} = h_2 \quad \text{for } h_i \in H_i, i = 1, 2.$$

Moreover, for any $x, y \in H_1$ and $u, v \in H_2$, we have

$$\begin{aligned}
\left\langle j_{H_1}(x), \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle_N &= \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, N\begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle = \langle x, N_1 y\rangle = \left\langle x, j_{H_1}^{-1}\begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle_{N_1}, \\
\left\langle j_{H_2}(u), \begin{pmatrix} -av \\ v \end{pmatrix} \right\rangle_N &= \left\langle \begin{pmatrix} -au \\ u \end{pmatrix}, N\begin{pmatrix} -av \\ v \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -au \\ u \end{pmatrix}, \begin{pmatrix} 0 \\ S(N)v \end{pmatrix} \right\rangle \\
&= \langle u, S(N)v\rangle = \left\langle u, j_{H_2}^{-1}\begin{pmatrix} -av \\ v \end{pmatrix} \right\rangle_{S(N)},
\end{aligned}$$

which means that j_{H_1} and j_{H_2} both are unitary operators. Let $j_{H_1} \oplus j_{H_2} : (H_1)_{N_1} \oplus (H_2)_{S(N)} \to X_1 \oplus X_2$ be the associated unitary operator defined by

$$(j_{H_1} \oplus j_{H_2})\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} j_{H_1}(h_1) \\ j_{H_2}(h_2) \end{pmatrix} = \begin{pmatrix} h_1 \\ 0 \\ --- \\ -ah_2 \\ h_2 \end{pmatrix} \quad \text{for } h_i \in H_i, i = 1, 2.$$

Then clearly, $j_{H_1}^{-1} \oplus j_{H_2}^{-1}$ is the adjoint operator of $j_{H_1} \oplus j_{H_2} \in \mathscr{L}((H_1)_{N_1} \oplus (H_2)_{S(N)}, X_1 \oplus X_2)$.

Let W be the positive definite element of $\mathscr{L}(H_1 \oplus H_2)$ defined by (5.7). Then clearly, the weighted space $(H_1 \oplus H_2)_W$ equals $(H_1)_{N_1} \oplus (H_2)_{S(N)}$. Suppose now H_3 is an additional Hilbert $\mathfrak{A}$-module, $M \in \mathscr{L}(H_3)$ is a positive definite element, and $(H_3)_M$ is the associated weighted space. Let $A = (A_1, A_2) \in \mathscr{L}(H_1 \oplus H_2, H_3)$ with $A_1 \in \mathscr{L}(H_1, H_3)$ and $A_2 \in \mathscr{L}(H_2, H_3)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
(H_1 \oplus H_2)_W = (H_1)_{N_1} \oplus (H_2)_{S(N)} \arrow[rr, "\lambda_X^{-1} \circ (j_{H_1} \oplus j_{H_2})"] \arrow[dr, "D"'] & & X = (H_1 \oplus H_2)_N \arrow[dl, "A"] \\
& (H_3)_M &
\end{tikzcd}$$

where $D = (D_1, D_2) = A \circ \lambda_X^{-1} \circ (j_{H_1} \oplus j_{H_2})$ with

$$D_1 = A_1 \quad \text{and} \quad D_2 = A_2 - A_1 N_1^{-1} L. \tag{5.11}$$

Since λ_X and $j_{H_1} \oplus j_{H_2}$ are unitary operators, we have

$$A_{MN}^{\dagger} = \lambda_X^{-1} \circ (j_{H_1} \oplus j_{H_2}) \circ D_{MW}^{\dagger}. \tag{5.12}$$

So if we let $D_{MW}^{\dagger} = \begin{pmatrix} (D_{MW}^{\dagger})_{11} \\ (D_{MW}^{\dagger})_{21} \end{pmatrix}$ with $(D_{MW}^{\dagger})_{11} \in \mathscr{L}((H_3)_M, (H_1)_{N_1})$, and $(D_{MW}^{\dagger})_{21} \in \mathscr{L}((H_3)_M, (H_2)_{S(N)})$, then by (5.12) we conclude that $A_{MN}^{\dagger} = \begin{pmatrix} (A_{MN}^{\dagger})_{11} \\ (A_{MN}^{\dagger})_{21} \end{pmatrix}$ with $(A_{MN}^{\dagger})_{11} \in \mathscr{L}(H_3, H_1)$ and $(A_{MN}^{\dagger})_{21} \in \mathscr{L}(H_3, H_2)$, such that

$$(A_{MN}^{\dagger})_{11} = (D_{MW}^{\dagger})_{11} - N_1^{-1} L (D_{MW}^{\dagger})_{21}, \tag{5.13}$$
$$(A_{MN}^{\dagger})_{21} = (D_{MW}^{\dagger})_{21}. \tag{5.14}$$

Note that $(H_1)_{N_1}, (H_2)_{S(N)}$ and $(H_3)_M$ all are Hilbert $\mathfrak{A}$-modules, and the weighted Moore–Penrose inverse $(D_1)_{MN_1}^{\dagger}$ of $D_1 \in \mathscr{L}(H_1, H_3)$ is exactly the regular Moore–Penrose inverse of D_1, where D_1 is considered as an element of $\mathscr{L}((H_1)_{N_1},(H_3)_M)$. Furthermore, by (5.2) we know that the adjoint operator $D_1^\#$ of $D_1 \in \mathscr{L}((H_1)_{N_1},(H_3)_M)$ equals $N_1^{-1}D_1^*M \in \mathscr{L}(H_3,H_1)$. Since Theorem 4.6 is valid for any Hilbert $\mathfrak{A}$-module operators, we can apply it to get a concrete expression for $D_{MW}^\dagger$, and then use formulae (5.13) and (5.14) to obtain an expression for $A_{MN}^\dagger$.

5.2. A new formula for the weighted Moore–Penrose inverse $(A_1,A_2)_{MN}^\dagger$

Theorem 5.1. *Let $A=(A_1,A_2)\in\mathscr{L}(H_1\oplus H_2,H_3)$ be a partitioned operator defined by (1.7) for some $A_1\in\mathscr{L}(H_1,H_3)$ and $A_2\in\mathscr{L}(H_2,H_3)$. Let $M\in\mathscr{L}(H_3)$ be a positive definite element and $N\in\mathscr{L}(H_1\oplus H_2)$ be a positive definite element decomposed as (5.6). Let $a=N_1^{-1}L, S(N)=N_2-L^*a, D_1,D_2$ be defined by (5.11). If $(D_1)_{MN_1}^\dagger,(D_2)_{MS(N)}^\dagger$ and $C_{MS(N)}^\dagger$ all exist, then*

$$(A_1,A_2)_{MN}^\dagger=\begin{pmatrix}(A_1)_{MN_1}^\dagger-(\Sigma+a)\Omega\\ \Omega\end{pmatrix}, \tag{5.15}$$

where

$$C=(I_{H_3}-A_1(A_1)_{MN_1}^\dagger)A_2,\quad \Sigma=(A_1)_{MN_1}^\dagger(A_2-A_1a), \tag{5.16}$$

$$Y=(I_{H_2}-C_{MS(N)}^\dagger C)S(N)^{-1}, \tag{5.17}$$

$$\Omega=(I_{H_2}+Y\Sigma^*N_1\Sigma)^{-1}(Y\Sigma^*N_1\cdot(A_1)_{MN_1}^\dagger+C_{MS(N)}^\dagger). \tag{5.18}$$

Proof. For simplicity, the identity operator on a Hilbert $\mathfrak{A}$-module is denoted simply by I. Since $A_1(A_1)_{MN_1}^\dagger A_1=A_1$, by (5.11) we get

$$(I-D_1(D_1)_{MN_1}^\dagger)D_2=(I-A_1(A_1)_{MN_1}^\dagger)(A_2-A_1a)=C\in\mathscr{L}((H_2)_{S(N)},(H_3)_M).$$

Therefore, by Theorem 4.6 we conclude that

$$D_{MW}^\dagger=(D_1,D_2)_{MW}^\dagger=\begin{pmatrix}(D_1)_{MN_1}^\dagger-(D_1)_{MN_1}^\dagger D_2B_2\\ B_2\end{pmatrix}, \tag{5.19}$$

where

$$K_1=(D_1)_{MN_1}^\dagger D_2(I-C_{MS(N)}^\dagger C)\in\mathscr{L}((H_2)_{S(N)},(H_1)_{N_1}), \tag{5.20}$$

$$B_2=(I+K_1^\#(D_1)_{MN_1}^\dagger D_2)^{-1}(K_1^\#(D_1)_{MN_1}^\dagger+C_{MS(N)}^\dagger). \tag{5.21}$$

Note that

$$D_2^\#=(-A_1a+A_2)^\#=S(N)^{-1}(-A_1a+A_2)^*M, \tag{5.22}$$

and

$$((D_1)_{MN_1}^\dagger)^\#=((A_1)_{MN_1}^\dagger)^\#=M^{-1}((A_1)_{MN_1}^\dagger)^*N_1. \tag{5.23}$$

By (5.20)–(5.23) and (5.11) we get

$$\begin{aligned}K_1^\#(D_1)_{MN_1}^\dagger&=(I-C_{MS(N)}^\dagger C)^\#D_2^\#((D_1)_{MN_1}^\dagger)^\#(D_1)_{MN_1}^\dagger\\&=(I-C_{MS(N)}^\dagger C)S(N)^{-1}(-A_1a+A_2)^*((A_1)_{MN_1}^\dagger)^*\cdot N_1\cdot(A_1)_{MN_1}^\dagger\\&=Y\Sigma^*N_1\cdot(A_1)_{MN_1}^\dagger.\end{aligned} \tag{5.24}$$

It follows from (5.24), (5.11) and (5.16) that

$$I+K_1^\#(D_1)_{MN_1}^\dagger D_2=I+Y\Sigma^*N_1\Sigma. \tag{5.25}$$

Combining (5.21), (5.25), (5.24) and (5.18) we conclude that $B_2 = \Omega$. Therefore, by (5.19), (5.11) and (5.16) we get

$$(D_{MW}^{\dagger})_{11} = (A_1)_{MN_1}^{\dagger} - \Sigma\Omega, \quad (D_{MW}^{\dagger})_{21} = \Omega. \tag{5.26}$$

Formula (5.15) then follows from (5.13), (5.14) and (5.26). □

The main result of [7] turns out to be a special case of Theorem 5.1:

Corollary 5.2 (*cf.* [7, Result 3.1]). *Let m and $n(n \geqslant 2)$ be two natural numbers, $H_1 = \mathbb{C}^{n-1,1}, H_2 = \mathbb{C}, H_3 = \mathbb{C}^{m,1}, A_1 \in \mathbb{C}^{m,n-1}$ and $A_2 \in \mathbb{C}^{m,1}$. Let $M \in \mathbb{C}^{m,m}$ be a positive definite matrix and $N = \begin{pmatrix} N_1 & L \\ L^* & N_2 \end{pmatrix} \in \mathbb{C}^{n,n}$ be a positive definite matrix with $N_1 \in \mathbb{C}^{n-1,n-1}$ and $N_2 \in \mathbb{C}$. Then*

$$(A_1, A_2)_{MN}^{\dagger} = \begin{pmatrix} (A_1)_{MN_1}^{\dagger} - (\Sigma + N_1^{-1}L)\Omega \\ \Omega \end{pmatrix}, \tag{5.27}$$

where

$$\Sigma = (A_1)_{MN_1}^{\dagger}(A_2 - A_1 N_1^{-1}L), \tag{5.28}$$

$$q = \begin{pmatrix} (A_1)_{MN_1}^{\dagger} A_2 + (I_{H_1} - (A_1)_{MN_1}^{\dagger} A_1) N_1^{-1} L \\ -1 \end{pmatrix}, \tag{5.29}$$

$$C = (I_{H_3} - A_1 (A_1)_{MN_1}^{\dagger}) A_2, \quad U = \begin{pmatrix} (A_1)_{MN_1}^{\dagger} \\ 0_{1\times m} \end{pmatrix}, \tag{5.30}$$

$$\Omega = \begin{cases} C_{MS(N)}^{\dagger}, & \text{if} \quad C \neq 0 \\ \frac{1}{q^* N q} q^* N U, & \text{if} \quad C = 0 \end{cases}. \tag{5.31}$$

Proof. Let $a = N_1^{-1}L$. Note that the matrix C defined by (5.30) is a column vector of m components and $S(N) = N_2 - L^*a$ is a complex number, the conclusion $C_{MS(N)}^{\dagger} C = 1$ or $C_{MS(N)}^{\dagger} C = 0$ follows since $CC_{MS(N)}^{\dagger} C = C$ and $C_{MS(N)}^{\dagger} C$ is a complex number.

Case 1: $C \neq 0$. In this case, the operator Y defined by (5.17) is zero, hence $\Omega = C_{MS(N)}^{\dagger}$ by (5.18).

Case 2: $C = 0$. In this case, by (5.17) and (5.18) we have

$$\begin{aligned} \Omega &= (1 + S(N)^{-1}\Sigma^* N_1 \cdot \Sigma)^{-1} S(N)^{-1} \Sigma^* N_1 (A_1)_{MN_1}^{\dagger} \\ &= \frac{1}{S(N) + \Sigma^* N_1 \Sigma} \Sigma^* N_1 \cdot (A_1)_{MN_1}^{\dagger}. \end{aligned} \tag{5.32}$$

On the other hand, we have $q = \binom{\Sigma + a}{-1}$, so

$$\begin{aligned} q^* N q &= (\Sigma^* + a^*, -1) \begin{pmatrix} N_1 & L \\ L^* & N_2 \end{pmatrix} \begin{pmatrix} \Sigma + a \\ -1 \end{pmatrix} \\ &= (\Sigma^* N_1, \Sigma^* L + a^* L - N_2) \begin{pmatrix} \Sigma + a \\ -1 \end{pmatrix} = S(N) + \Sigma^* N_1 \Sigma, \end{aligned} \tag{5.33}$$

and

$$q^* N U = (\Sigma^* N_1, \Sigma^* L + a^* L - N_2) \begin{pmatrix} (A_1)_{MN_1}^{\dagger} \\ 0 \end{pmatrix} = \Sigma^* N_1 \cdot (A_1)_{MN_1}^{\dagger}. \tag{5.34}$$

The conclusion then follows from (5.32)–(5.34). □

We give a numerical example as follows:

Example 5.1. Let

$$A=\begin{pmatrix}1&0&3&0\\0&-2&0&1\\1&0&2&0\\0&0&-1&0\end{pmatrix},\quad M=\begin{pmatrix}1&0&1&0\\0&2&0&0\\1&0&3&0\\0&0&0&1\end{pmatrix}\quad\text{and}\quad N=\begin{pmatrix}1&1&0&0\\1&2&0&0\\0&0&1&0\\0&0&0&2\end{pmatrix}.$$

According to Theorem 5.1, two methods can be taken into consideration to compute the weighted Moore–Penrose inverse $A^{\dagger}_{MN}$:

1. Let

$$A_1=\begin{pmatrix}1&0\\0&-2\\1&0\\0&0\end{pmatrix},\quad A_2=\begin{pmatrix}3&0\\0&1\\2&0\\-1&0\end{pmatrix},\quad N_1=\begin{pmatrix}1&1\\1&2\end{pmatrix},$$

$$a=L=\begin{pmatrix}0&0\\0&0\end{pmatrix}\quad\text{and}\quad S(N)=\begin{pmatrix}1&0\\0&2\end{pmatrix}.$$

Then

$$(A_1)^{\dagger}_{MN_1}=\begin{pmatrix}\frac{1}{3}&0&\frac{2}{3}&0\\0&-\frac{1}{2}&0&0\end{pmatrix},\quad \Sigma=\begin{pmatrix}\frac{7}{3}&0\\0&-\frac{1}{2}\end{pmatrix},\quad C=\begin{pmatrix}\frac{2}{3}&0\\0&0\\-\frac{1}{3}&0\\-1&0\end{pmatrix},$$

$$C^{\dagger}_{MS(N)}=\begin{pmatrix}\frac{1}{4}&0&-\frac{1}{4}&-\frac{3}{4}\\0&0&0&0\end{pmatrix}\quad\text{and}\quad \Omega=\begin{pmatrix}\frac{1}{4}&0&-\frac{1}{4}&-\frac{3}{4}\\\frac{1}{20}&\frac{1}{5}&-\frac{1}{4}&-\frac{7}{20}\end{pmatrix},$$

so

$$A^{\dagger}_{MN}=\begin{pmatrix}-\frac{1}{4}&0&\frac{5}{4}&\frac{7}{4}\\\frac{1}{40}&-\frac{2}{5}&-\frac{1}{8}&-\frac{7}{40}\\\frac{1}{4}&0&-\frac{1}{4}&-\frac{3}{4}\\\frac{1}{20}&\frac{1}{5}&-\frac{1}{4}&-\frac{7}{20}\end{pmatrix}.$$

2. Let

$$A_1=\begin{pmatrix}1&0&3\\0&-2&0\\1&0&2\\0&0&-1\end{pmatrix},\quad A_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad N_1=\begin{pmatrix}1&1&0\\1&2&0\\0&0&1\end{pmatrix},$$

$$a=L=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\text{and}\quad S(N)=2.$$

Then

$$(A_1)^{\dagger}_{MN_1}=\begin{pmatrix}-\frac{1}{4}&0&\frac{5}{4}&\frac{7}{4}\\0&-\frac{1}{2}&0&0\\\frac{1}{4}&0&-\frac{1}{4}&-\frac{3}{4}\end{pmatrix}$$

and

$$\Sigma=\begin{pmatrix}0\\-\frac{1}{2}\\0\end{pmatrix},\quad C=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\quad \Omega=\begin{pmatrix}\frac{1}{20}&\frac{1}{5}&-\frac{1}{4}&-\frac{7}{20}\end{pmatrix},$$

so

$$A^{\dagger}_{MN} = \begin{pmatrix} -\frac{1}{4} & 0 & \frac{5}{4} & \frac{7}{4} \\ \frac{1}{40} & -\frac{2}{5} & -\frac{1}{8} & -\frac{7}{40} \\ \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} \\ \frac{1}{20} & \frac{1}{5} & -\frac{1}{4} & -\frac{7}{20} \end{pmatrix}.$$

Acknowledgments

The author thanks Professor Chi-Kwong Li for his help. He also thanks the referee for many helpful comments and suggestions, which have improved this paper significantly.

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